Q/ Show that the sequence is convergent

$$\langle x_n \rangle = \left\langle \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right\rangle \quad \forall n \in N$$

Sol^n

Here $x_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots \frac{1}{(n+1)+(n+1)}$

Then $x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$\Rightarrow x_{n+1} - x_n \geq \frac{1}{2n+1} + \frac{1}{2n+2} \quad \forall n \in N$

$\Rightarrow x_{n+1} - x_n > 0 \quad \forall n \in N$

$\Rightarrow \langle x_n \rangle$ is an increasing sequence.

Again $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$

$= \frac{n}{n} = 1$

$\Rightarrow x_n < 1 \quad \forall n \in N$

$\Rightarrow \langle x_n \rangle$ is bounded from above.

$\Rightarrow \langle x_n \rangle$ is increasing and bounded from above

$\Rightarrow \langle x_n \rangle$ is convergent.

Q/. Prove that the following sequences converges to 2.

Sol^n. (i) $\langle x_n \rangle$, where $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$

(ii) $\langle x_n \rangle$, where $x_1 = 1$, $x_n = \sqrt{2 + x_{n-1}}$

Sol^n: $x_1 = \sqrt{2}$ ——(i), $x_2 = \sqrt{2x_1}$
$= \sqrt{2\sqrt{2}}$

$\Rightarrow x_2 = \sqrt{2\sqrt{2}}$ ——— (ii)

Take $\sqrt{2} > 1$

$\Rightarrow 2\sqrt{2} > 2 \Rightarrow \sqrt{2\sqrt{2}} > \sqrt{2}$

$\Rightarrow x_2 > x_1$

or $x_1 < x_2$ ——— (iii)

Now, if $x_{l-1} < x_l$

$\Rightarrow \sqrt{2x_{l-1}} < \sqrt{2x_l}$

$\Rightarrow x_l < x_{l+1}$ ——— (iv)

Then by the induction method, we can show that $x_n < x_{n+1}$ ——— (v)

~~Now~~ $\Rightarrow \langle x_n \rangle$ is an increasing sequence.

Again $x_1 < 2$, then let $x_l < 2$

$\cancel{\Rightarrow 2x_l}$ $x_{l+1} = \sqrt{2x_l}$

$\therefore x_{l+1} \leq \sqrt{2.2}$

$\Rightarrow x_{l+1} < 2$

$\Rightarrow x_n < 2 \quad \forall n \in N$

$\Rightarrow \langle x_n \rangle$ is bounded from above

Therefore, $\langle x_n \rangle$ is monotonically increasing & bounded from above

$\Rightarrow \langle x_n \rangle$ is convergent &

let $\lim_{n\to\infty} x_n = l$

then, take $x_{n+1} = \sqrt{2x_n}$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = \sqrt{2.\lim_{n\to\infty} x_n}$

$l = \sqrt{2l}$

$\Rightarrow l^2 - 2l = 0 \Rightarrow l(l-2) = 0$

$\Rightarrow$ Either $l = 0$ or $l = 2$

But l can't be zero as $x_1 = \sqrt{2}$ &

it is increasing

$\Rightarrow l = 2$

$\Rightarrow \langle x_n \rangle$ converges to 2.

Q/ Prove that the sequence $\langle x_n \rangle$, where

$x_1 = \sqrt{7}$, $x_{n+1} = \sqrt{7 + x_n}$ converges to

positive root of the equation $x^2 - x - 7 = 0$

Sol^n. Here $x_1 = \sqrt{7}$ given

$x_{n+1} = \sqrt{7 + x_n}$

$\Rightarrow \quad x_2 = \sqrt{7 + x_1}$

$\Rightarrow \quad x_2 = \sqrt{7 + \sqrt{7}}$

$\Rightarrow \quad x_2 > x_1$ —— (i)

similarly $x_3 = \sqrt{7 + x_2}$

$\Rightarrow x_3 > x_2$ —— (ii)

$\Rightarrow \quad x_{l+1} > x_l$

$\Rightarrow \sqrt{7 + x_{l+1}} > \sqrt{7 + x_l}$

$\Rightarrow$

$x_{l+2} > x_{l+1}$

$\Rightarrow$ By induction method $x_{n+1} > x_n \; \forall n \in N.$

$\Rightarrow$ Series is increasing. ——— (i)

Again $\quad x_1 = \sqrt{7}$

< 7

& $x_2 = \sqrt{7 + \sqrt{7}}$

< 7

Similarly $x_n < 7 \quad \forall n \in N.$

Therefore, $\langle x_n \rangle$ is bounded from above ——— (ii)

$\Rightarrow \langle x_n \rangle$ is convergent.

Again let $\lim_{n \to \infty} x_n = l$

Then, $x_{n+1} = \sqrt{7 + x_n}$

$\Rightarrow \lim_{n \to \infty} x_{n+1} = \sqrt{7 + \lim_{n \to \infty} x_n}$

$\Rightarrow \quad l = \sqrt{7 + l}$

$\Rightarrow \quad l^2 - l - 7 = 0$

$\Rightarrow l = \dfrac{1 \pm \sqrt{1 + 4 \times 1 \times 7}}{2 \cdot 1} = \dfrac{1 \pm \sqrt{29}}{2}$

$\Rightarrow l = \dfrac{1 + \sqrt{29}}{2} \quad \because l > 0$